FX derivatives

Juergen Hakala, Robin Stuart, Tasos Hamosfakidis

Standard Chartered Bank
Quantitative Development Group

17th March 2008

- The FX market
- Smile models
- Calibration and backmapping

Volatility is quoted in delta. Which delta?

- Spot-delta vs Forward-delta
- In FX two currencies, could quote delta in either of the two currencies.
- In FX the premium can be payed in either currency, which will effect the delta.

ATM

- ATM = forward
- ATM = delta neutral straddle

Volatility is quoted in market instruments:

Risk reversal

- X% delta risk reversal
- Long X% delta call, short X% delta put.
- Quoted as the difference of the call and put volatility

Butterfly

- X% delta fly
- Long Y% delta call, long Y% delta put, short ATM
- Quoted as the difference of the average call and put volatility to the ATM vol

From risk reversal and butterfly

- $\text{call}_X = \text{ATM} + \frac{1}{2}\text{RR}_X + \text{Fly}_X$
- $\text{put}_X = \text{ATM} - \frac{1}{2}\text{RR}_X + \text{Fly}_X$

Simple construction, but is not market practice!

Market strangle
The average in volatility of two strike using the given 'common' butterfly volatility is given by the market:

- $$K^{Fly}_{call}(\sigma_{fly} + \sigma_{ATM}, \delta)$$
- $$K^{Fly}_{put}(\sigma_{fly} + \sigma_{ATM}, \delta)$$

Risk reversal

- $$K^{RR}_{call}(\sigma_{ATM} + \frac{1}{2}\sigma_{RR}, \delta)$$
- $$K^{RR}_{put}(\sigma_{ATM} - \frac{1}{2}\sigma_{RR}, \delta)$$

For small risk reversal and moderate maturities the simple construction is a good approximation.

The problem to be solved is not an interpolation problem anymore. The smile to be constructed has a single point as an anchor: the ATM point. The other points are not explicitly given, but as constraints.
Risk reversal constraint:

$$P_{call}(K^{RR}_{call}, \sigma(K^{RR}_{call})) - P_{put}(K^{RR}_{put}, \sigma(K^{RR}_{put})) = P_{RR}$$

with

$$P_{RR} = P_{call}(K^{RR}_{call}, \sigma_{ATM} + \frac{1}{2}\sigma_{RR}) - P_{put}(K^{RR}_{put}, \sigma_{ATM} - \frac{1}{2}\sigma_{RR})$$

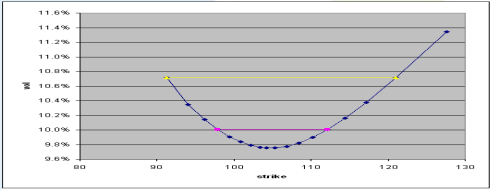
Strangle constraint:

$$P_{\text{call}}(K_{\text{call}}^{\text{Fly}}, \sigma(K_{\text{call}}^{\text{Fly}})) + P_{\text{put}}(K_{\text{put}}^{\text{Fly}}, \sigma(K_{\text{put}}^{\text{Fly}})) = P_{\text{Fly}}$$

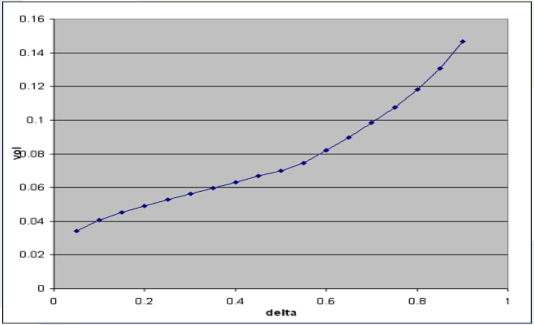
with

$$P_{\text{Fly}} = P_{\text{call}}(K_{\text{call}}^{\text{Fly}}, \sigma_{\text{ATM}} + \sigma_{\text{Fly}}) + P_{\text{put}}(K_{\text{put}}^{\text{Fly}}, \sigma_{\text{ATM}} + \sigma_{\text{Fly}})$$

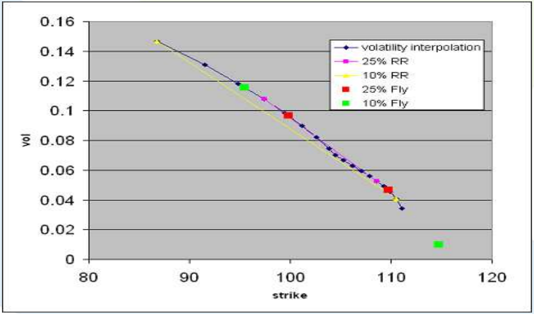
Symmetric smile



Difficulties for skewed smile



Difficulties for skewed smile



- one day forward volatility is not the same for all days
- weekends and global holidays are non-volatile days
- local holidays are non-volatile days for local currencies
- economic indicator and number publications create additional variance

- smile–consistency
- control forward–smile dynamics
- reproduce market prices

- Vega–hedge motivated
-

$$P = P_{ATM} + \Theta_{S\sigma} C_{vanna} \qquad \partial_{S\sigma} P + \Theta_{\sigma\sigma} C_{volga} \qquad \partial_{\sigma\sigma} P$$

The vanna cost C_{vanna} and volga cost C_{volga} is usually derived from 25Δ risky and fly. The factors $\Theta_{S\sigma}, \Theta_{\sigma\sigma}$ are often chosen as the expected survival time in percent of the maturity. There is some justification for OT to do so.

- Dupire's local volatility
- Parametric local volatility

$$
\begin{aligned}
\frac{dS}{S} &= (r^d(t) - r^f(t))dt + \sigma_i(S,t)dW \\
\sigma_i(S_t,t) &= f(S_0, S_t, t)
\end{aligned}
$$

Standard Chartered

Parametric local volatility

$$
\begin{aligned}
f_1(S_0, S_t, t) &= f(LR = \ln(S_t/S_0), t) = c(t) + b(t)LR + a(t)LR^2 \\
f_2(S_0, S_t, t) &= c_{-2}(t)LR^{-2} + c-1(t)LR^{-1} + c_0(t) \\
&\quad + c_1(t)LR + c_2(t)LR^2
\end{aligned}
$$

other forms are available and are more adapted to other asset classes.

Most popular is the Heston model

$$\begin{aligned}\frac{dS}{S} &= (r^d(t) - r^f(t))dt + \sigma_i(t)dW \\ d\sigma^2 &= \kappa(\bar{\sigma}^2 - \sigma^2(t))dt + \xi\sigma_i dV\end{aligned}$$

with $< dW, dV >= \rho dt$. A closed form solution exists.

Smile models

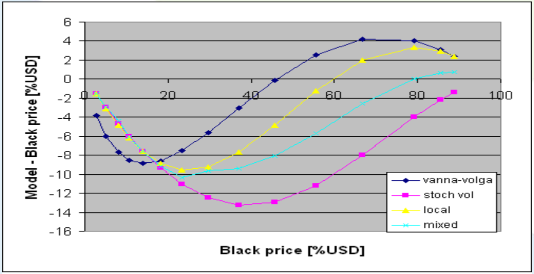
Mixed stoch/local vol

Introduce mixing of stochastic volatility model and local volatility model.

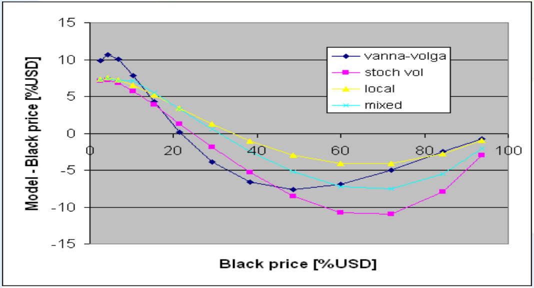
$$
\begin{aligned}
\frac{dS}{S} &= (r^d(t) - r^f(t))dt + \sigma_i(t) f_\lambda(S_t, t) dW \\
d\sigma^2 &= \kappa(\bar{\sigma}^2 - \sigma^2(t))dt + \xi_\lambda \sigma_i dV
\end{aligned}
$$

with $< dW, dV >= \rho dt$ and the mixing factor λ. For $\lambda = 1$ full Heston model, and $f = 1$, for $\lambda = 0$ full local vol model with local vol function and $\xi_\lambda = 0$.

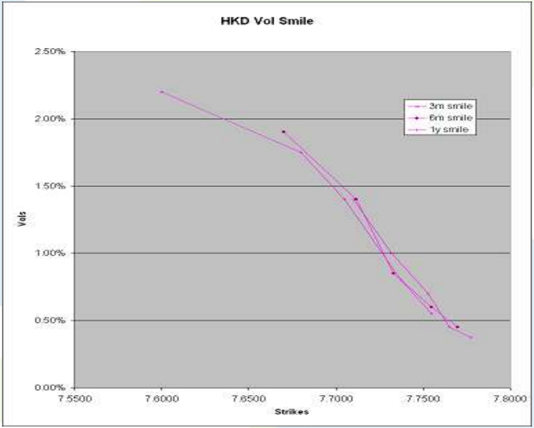
USD/JPY OT 1y upside



USD/JPY OT 1y downside



Pegged and managed currencies



Standard Chartered

- jump diffusion models
  - distribution of jump–size [Kou, Merton]
  - stochastic–volatility and jumps
- stochastic skew model

- jump diffusion models
  - distribution of jump–size [Kou, Merton]
  - stochastic–volatility and jumps
- stochastic skew model skew in some markets very volatile, RR changes sign

- Avoid Schoenbucher 'mean fleeing' dynamics close to maturity by a finite forward–vol model?
- Is it really necessary when we can use long time stepping for Heston model in Monte Carlo?

- Sparse input
- Least mean squares method in volatility, prices or weighted prices

Assuming that calibration solves a least mean squares problem
The objective function for non linear least square optimisation for instruments M_i with target price O_i

$$\sum_{i=1}^{M} (M_i(P) - O_i)^2 . \tag{1}$$

The minimization of the objective function will lead to the local normal equation to be fulfilled for all calibration parameters $P_1, \cdots, P_N$.

$$\sum_{i=1}^{M} \partial_j M_i(P) \left(M_i(P) - O_i\right) = 0. \tag{2}$$

The sensitivity of the price of a product to the change in parameter β is given by

$$dV = \partial_\beta V d\beta + \sum_{k=1}^{N} \partial_{\alpha^k} V \partial_\beta \alpha^k \quad d\beta, \tag{3}$$

One way to get that impact on the calibration is the finite difference on the parameter and a re–calibration.

The expansion of the price with respect to calibrated parameter α^k around the solution of the calibration procedure $\alpha_0{}^k$ yields the

$$M_i(\bar{\alpha}) = M_i(\alpha_0) + \sum_{j=1}^{N} \partial_{\alpha^j} M_i(\bar{\alpha}^j - \alpha_0{}^j) + O(2) \tag{4}$$

The LS eq. is rewritten (with $\alpha^j = \bar{\alpha}^j - \alpha_0{}^j$) as

$$\sum_{i=1}^{M} \left(M_i(\alpha_0) + \sum_{j=1}^{N} \partial_{\alpha^j} M_i \alpha^j - O_i \right) = 0. \tag{5}$$

Standard Chartered

Normal equations

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\alpha^{j}\partial_{\alpha^{j}}M_{i}(\alpha_{0})\partial_{\alpha^{k}}M_{i}(\alpha_{0})=\sum_{i=1}^{M}\left(M_{i}(\alpha_{0})-O_{i}\right)\partial_{\alpha^{k}}M_{i}(\alpha_{0}). \tag{6}$$

since $\alpha^{j}=0$ is a solution by definition the constraint is

$$\sum_{i=1}^{M}M_{i}(\alpha_{0})\partial_{\alpha^{k}}M_{i}(\alpha_{0})=\sum_{i=1}^{M}O_{i}\partial_{\alpha^{k}}M_{i}(\alpha_{0}) \tag{7}$$

Standard Chartered

The required sensitivity is obtained by solving the system of equations

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\partial_{\beta}\alpha^{j}\partial_{\alpha^{j}}M_{i}\partial_{\alpha^{k}}M_{i} = \sum_{i=1}^{M}\left(M_{i}-O_{i}\right)\partial_{\alpha^{k}\beta}M_{i} + \sum_{i=1}^{M}\left(\partial_{\beta}M_{i}-\partial_{\beta}O_{i}\right)\partial_{\alpha^{k}}M_{i}. \tag{8}$$

This is a very useful equation, that can be used in many cases. It simplifies for specific cases as well.

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\partial_\beta\alpha^j\partial_{\alpha^j}M_i\partial_{\alpha^k}M_i = -\partial_{\alpha^k}M_i. \tag{9}$$

- sensitivity to input prices O_i

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\partial_\beta\alpha^j\partial_{\alpha^j}M_i\partial_{\alpha^k}M_i = -\partial_\beta O_i\partial_{\alpha^k}M_i. \tag{9}$$

- sensitivity to input prices O_i or vols

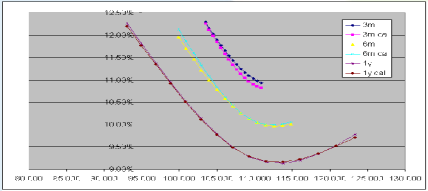
$$\sum_{i=1}^{M}\sum_{j=1}^{N}\partial_\beta\alpha^j\partial_{\alpha^j}M_i\partial_{\alpha^k}M_i = -\partial_\beta O_i\partial_{\alpha^k}M_i. \tag{9}$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\partial_\beta\alpha^j\partial_{\alpha^j}M_i\partial_{\alpha^k}M_i = -\sum_{i=1}^{M}\partial_\beta\sigma\partial_\sigma O_i\partial_{\alpha^k}M_i. \tag{10}$$

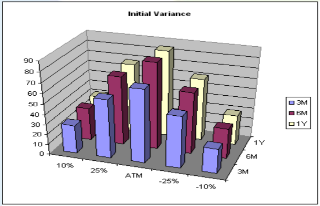
- sensitivity to input prices O_i or vols
- sensitivity to ATM, RR and Fly as a strategy

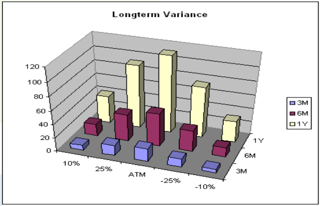
Standard Chartered

- USDJPY market data 1y [modified]
- Calibrationset maturities 3m, 6m, 1y
- equivalent strikes to 10%, 25%, ATM, -25%, -10%

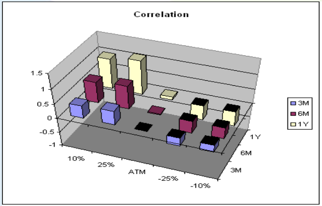


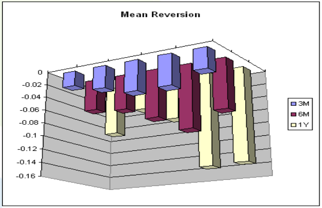
Initial variance and long term variance

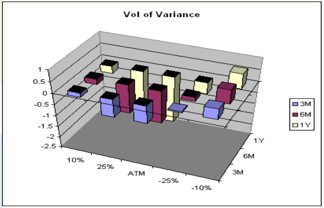




Correlation variance and Mean reversion







Risk for vol of variance

Estimated calibration change and actual change [scaled by 10000] for parallel move of the vol surface

	estimated	actual
initial var	5.19	3.84
longterm var	3.77	3.41
correlation	44.1	49.5
meanreversion	30.0	-625.07
vol of var	-29.95	-42.26

Estimated calibration change and actual change [scaled by 10000] for risk reversal

	estimated	actual
initial var	-0.3	0.6
longterm var	-1.7	-0.5
correlation	491	793
meanreversion	56	58
vol of var	-9.61	6.16

Estimated calibration change and actual change [scaled by 10000] for butterfly

	estimated	actual
initial var	0.69	1.2
longterm var	0.96	0.65
correlation	79	141
meanreversion	-314	-598
vol of var	91	123

Conclusion

- Get the market convention for FX correct is not a trivial task

Conclusion

- Get the market convention for FX correct is not a trivial task
- Selection criteria for a smile model are different for different markets, in particular for non G7 markets

Conclusion

- Get the market convention for FX correct is not a trivial task
- Selection criteria for a smile model are different for different markets, in particular for non G7 markets
- Back–mapping of sensitivities improves speed and stability of the greeks when calibration is involved.

Conclusion

- Get the market convention for FX correct is not a trivial task
- Selection criteria for a smile model are different for different markets, in particular for non G7 markets
- Back–mapping of sensitivities improves speed and stability of the greeks when calibration is involved.
- Back–mapping of sensitivities improves the understanding of the model