Part I. Market: Products and Basics

Chapter 1
Vanilla Options

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1.1 Model and payoff

We consider the model geometric Brownian motion

$$dS_t = (r_d - r_f)S_t\,dt + \sigma S_t\,dW_t \tag{1.1}$$

The parameters r_d, r_f and σ are called the domestic interest rate, the foreign interest rate and the volatility respectively.

Applying Ito's rule to $\ln S_t$ yields the following solution for the process S_t

$$S_t = S_0 \exp\left\{(r_d - r_f - \frac{1}{2}\sigma^2)t + \sigma W_t\right\} \tag{1.2}$$

which shows that S_t is log-normally distributed, more precisely, $\ln S_t$ is normal with mean $\ln S_0 + \left(r_d - r_f - \frac{1}{2}\sigma^2\right)t$ and variance $\sigma^2 t$. Further model assumptions are

1. There is no arbitrage.
2. Trading is frictionless, no transaction cost.
3. Any position can be taken at any time, short, long, arbitrary fraction, no liquidity constraints.

The payoff for a vanilla option (European put or call) is given by

$$F = [\phi(S_T - K)]^+ \tag{1.3}$$

where the contractual parameters are strike K, expiration time T and type ϕ, a binary variable which takes the value $+1$ in the case of a call and -1 in the case of a put. The symbol x^+ denotes the positive part of x, ie, $x^+ \triangleq \max(0, x) \triangleq 0 \vee x$.

1.2 Value

In the Black–Scholes model the value of the payoff F at time t if the spot is at x is denoted by $v(t, x)$ and can be computed either as the solution of the Black–Scholes Partial Differential Equation (PDE)

$$v_t - r_d v + (r_d - r_f) x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0 \quad (1.4)$$

$$v(T, x) = F \quad (1.5)$$

or equivalently (Feynman–Kac Theorem) as a discounted expected value

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = e^{-r_d \tau} \mathbb{E}[F] \quad (1.6)$$

This is why basic financial engineering is mostly concerned with solving PDEs or computing expectations (numerical integration). The result is the Black–Scholes formula

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = \phi e^{-r_d \tau} [f \mathcal{N}(\phi d_+) - K \mathcal{N}(\phi d_-)] \quad (1.7)$$

1.2.1 *Abbreviations*

- x: current price of the underlying
- $\tau \triangleq T - t$
- $f \triangleq \mathbb{E}[S_T | S_t = x] = x e^{(r_d - r_f)\tau}$: forward price of the underlying
- $\theta_\pm \triangleq \dfrac{r_d - r_f}{\sigma} \pm \dfrac{\sigma}{2}$
- $d_\pm \triangleq \dfrac{\ln \frac{x}{K} + \sigma \theta_\pm \tau}{\sigma \sqrt{\tau}} = \dfrac{\ln \frac{f}{K} \pm \frac{\sigma^2}{2} \tau}{\sigma \sqrt{\tau}}$
- $n(t) \triangleq \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = n(-t)$
- $\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\, dt = 1 - \mathcal{N}(-x)$

The Black–Scholes formula can be derived using the integral representation of Equation (1.6)

$$\begin{aligned} v &= e^{-r_d \tau} \mathbb{E}[F] \\ &= e^{-r_d \tau} \mathbb{E}[[\phi(S_T - K)]^+] \\ &= e^{-r_d \tau} \int_{-\infty}^{+\infty} \left[\phi\left(x e^{(r_d - r_f - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} y} - K\right)\right]^+ n(y)\, dy \end{aligned} \quad (1.8)$$

Next, one has to deal with the positive part and then complete the square to get the Black–Scholes formula. A derivation based on the PDE can be done using results about the heat equation (see, eg Wilmott, Howison and Dewynne [14]).

1.2.2 *A note on the forward*

The forward price f is the strike which makes the time zero value of the forward contract

$$F = S_T - f \quad (1.9)$$

equal to zero. It follows that $f = \mathbb{E}[S_T] = x e^{(r_d - r_f)T}$, ie the forward price is the expected price of the underlying at time T in a risk-neutral (drift of the geometric Brownian motion is equal to cost of carry $r_d - r_f$) setup. The situation $r_d > r_f$ is called contango, and the situation $r_d < r_f$ is called backwardation. Note that in the Black–Scholes model the class of forward price curves is quite restricted. For example, no seasonal effects can be included. Note that the value of the forward contract after time zero is usually different from zero, and since one of the counterparties is always short, there may be risk of default of the short party. A futures contract prevents this dangerous affair: it is basically a forward contract, but the counterparties have to maintain margin accounts to ensure the amount of cash or commodity owed does not exceed a specified limit.

1.3 Greeks

Greeks are derivatives of the value function with respect to model and contract parameters. They are an important information for traders and have become standard information supplied by front-office systems. More details on relations among Greeks are presented in Chapter 13. For vanilla options we list some of them now.

(Spot) delta

$$\frac{\partial v}{\partial x} = \phi e^{-r_f\tau}\mathcal{N}(\phi d_+) \tag{1.10}$$

Forward delta

$$\frac{\partial v}{\partial f} = \phi e^{-r_d\tau}\mathcal{N}(\phi d_+) \tag{1.11}$$

Driftless delta

$$\phi\mathcal{N}(\phi d_+) \tag{1.12}$$

Gamma

$$\frac{\partial^2 v}{\partial x^2} = e^{-r_f\tau}\frac{n(d_+)}{x\sigma\sqrt{\tau}} \tag{1.13}$$

Speed

$$\frac{\partial^3 v}{\partial x^3} = -e^{-r_f\tau}\frac{n(d_+)}{x^2\sigma\sqrt{\tau}}\left(\frac{d_+}{\sigma\sqrt{\tau}} + 1\right) \tag{1.14}$$

Theta

$$\frac{\partial v}{\partial t} = -e^{-r_f\tau}\frac{n(d_+)x\sigma}{2\sqrt{\tau}} + \phi[r_f x e^{-r_f\tau}\mathcal{N}(\phi d_+) - r_d K e^{-r_d\tau}\mathcal{N}(\phi d_-)] \tag{1.15}$$

Charm

$$\frac{\partial^2 v}{\partial x\partial\tau} = -\phi r_f e^{-r_f\tau}\mathcal{N}(\phi d_+) + \phi e^{-r_f\tau}n(d_+)\frac{2(r_d - r_f)\tau - d_-\sigma\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}} \tag{1.16}$$

Colour

$$\frac{\partial^3 v}{\partial x^2\,\partial \tau} = -e^{-r_f \tau}\frac{n(d_+)}{2x\tau\sigma\sqrt{\tau}}\left[2r_f\tau + 1 + \frac{2(r_d - r_f)\tau - d_-\sigma\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}}d_+\right] \tag{1.17}$$

Vega

$$\frac{\partial v}{\partial \sigma} = xe^{-r_f\tau}\sqrt{\tau}n\,(d_+) \tag{1.18}$$

Volga

$$\frac{\partial^2 v}{\partial \sigma^2} = xe^{-r_f\tau}\sqrt{\tau}n(d_+)\frac{d_+ d_-}{\sigma} \tag{1.19}$$

Vanna

$$\frac{\partial^2 v}{\partial \sigma \partial x} = -e^{-r_f\tau}n(d_+)\frac{d_-}{\sigma} \tag{1.20}$$

Rho

$$\frac{\partial v}{\partial r_d} = \phi K\tau e^{-r_d\tau}\mathcal{N}(\phi d_-) \tag{1.21}$$

$$\frac{\partial v}{\partial r_f} = -\phi x\tau e^{-r_f\tau}\mathcal{N}(\phi d_+) \tag{1.22}$$

Dual Delta

$$\frac{\partial v}{\partial K} = -\phi e^{-r_d\tau}\mathcal{N}(\phi d_-) \tag{1.23}$$

Dual Gamma

$$\frac{\partial^2 v}{\partial K^2} = e^{-r_d\tau}\frac{n(d_-)}{K\sigma\sqrt{\tau}} \tag{1.24}$$

Dual Theta

$$\frac{\partial v}{\partial T} = -\frac{\partial v}{\partial t} \tag{1.25}$$

1.4 Identities

$$\frac{\partial d_\pm}{\partial \sigma} = -\frac{d_\mp}{\sigma} \tag{1.26}$$

$$\frac{\partial d_\pm}{\partial r_d} = \frac{\sqrt{\tau}}{\sigma} \tag{1.27}$$

$$\frac{\partial d_\pm}{\partial r_f} = -\frac{\sqrt{\tau}}{\sigma} \tag{1.28}$$

$$xe^{-r_f\tau}n(d_+) = Ke^{-r_d\tau}n(d_-) \tag{1.29}$$

$$\mathcal{N}(\phi d_-) = I\!P[\phi\, S_T \geqslant \phi K] \tag{1.30}$$

$$\mathcal{N}(\phi d_+) = I\!P\left[\phi S_T \leqslant \phi\,\frac{f^2}{K}\right] \tag{1.31}$$

1.4.1 *Put–call parity*

The put–call parity is the relationship

$$v(x, K, T, t, \sigma, r_d, r_f, +1) - v(x, K, T, t, \sigma, r_d, r_f, -1) = xe^{-r_f \tau} - Ke^{-r_d \tau} \tag{1.32}$$

which is just a more complicated way to write the trivial equation $x = x^+ - x^-$.

1.4.2 *Put–call delta parity*

$$\frac{\partial v(x, K, T, t, \sigma, r_d, r_f, +1)}{\partial x} - \frac{\partial v(x, K, T, t, \sigma, r_d, r_f, -1)}{\partial x} = e^{-r_f \tau} \tag{1.33}$$

In particular, we learn that the absolute value of a put delta and a call delta are not exactly adding up to one, but only to a positive number $e^{-r_f \tau}$. They add up to one approximately if either the time to expiration τ is short or if the foreign interest rate r_f is close to zero.

1.4.3 *Delta-symmetric strike*

While the choice $K = f$ produces identical values for call and put, we seek the strike $\check{K}$ which produces absolutely identical deltas (spot, forward or driftless). This condition implies $d_+ = 0$ and thus

$$\check{K} = fe^{\frac{\sigma^2}{2}T} \tag{1.34}$$

in which case the absolute delta is $e^{-r_f \tau}/2$. In particular, we learn, that always $\check{K} > f$, ie there can't be a put and a call with identical values *and* deltas. Note that the strike $\check{K}$ is usually chosen as the middle strike when trading a straddle or a butterfly. Similarly the dual delta symmetric strike $\hat{K} = fe^{-\frac{\sigma^2}{2}T}$ can be derived from the condition $d_- = 0$.

1.4.4 *Space-homogeneity*

We may wish to measure the value of the underlying in a different unit. This will obviously affect the option pricing formula as follows

$$av(x, K, T, t, \sigma, r_d, r_f, \phi) = v(ax, aK, T, t, \sigma, r_d, r_f, \phi) \text{ for all } a > 0 \tag{1.35}$$

Differentiating both sides with respect to a and then setting $a = 1$ yields

$$v = xv_x + Kv_K \tag{1.36}$$

Comparing the coefficients of x and K in Equations (1.7) and (1.36) leads to suggestive results for the delta v_x and dual delta v_K. This homogeneity is the reason behind the simplicity of the delta formulas, whose tedious computation can be saved in this way.

1.4.5 *Time-homogeneity*

We can perform a similar computation for the time-affected parameters and obtain the obvious equation

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = v\left(x, K, \frac{T}{a}, \frac{t}{a}, \sqrt{a}\sigma, ar_d, ar_f, \phi\right) \text{ for all } a > 0 \tag{1.37}$$

Differentiating both sides with respect to a and then setting $a = 1$ yields

$$0 = \tau v_t + \frac{1}{2}\sigma v_\sigma + r_d v_{r_d} + r_f v_{r_f} \tag{1.38}$$

Of course, this can also be verified by direct computation. The overall use of such equations is to generate double checking benchmarks when computing Greeks. These homogeneity methods can easily be extended to other more complex options.

1.4.6 *Put–call symmetry*

By put–call symmetry we understand the relationship (see Bates [2], Bates [3], Bowie [5] and Carr [6])

$$v(x, K, T, t, \sigma, r_d, r_f, +1) = \frac{K}{f} v\left(x, \frac{f^2}{K}, T, t, \sigma, r_d, r_f, -1\right) \tag{1.39}$$

The strike of the put and the strike of the call result in a geometric mean equal to the forward f. The forward can be interpreted as a geometric mirror reflecting a call into a certain number of puts. Note that for at-the-money options ($K = f$) the put–call symmetry coincides with the special case of the put–call parity where the call and the put have the same value.

1.4.7 *Rates symmetry*

Direct computation shows that the rates symmetry

$$\frac{\partial v}{\partial r_d} + \frac{\partial v}{\partial r_f} = -\tau v \tag{1.40}$$

holds for vanilla options. This relationship, in fact, holds for all European options and a wide class of path-dependent options as shown in Chapter 13.

1.4.8 *Foreign–domestic symmetry*

One can directly verify the relationship

$$\frac{1}{x} v(x, K, T, t, \sigma, r_d, r_f, \phi) = K v\left(\frac{1}{x}, \frac{1}{K}, T, t, \sigma, r_f, r_d, -\phi\right) \tag{1.41}$$

This equality can be viewed as one of the faces of put–call symmetry. The reason is that the value of an option can be computed both in a domestic as well as in a foreign scenario. We consider the example of S_t modelling the exchange rate of €/US\$. In New York, the call option $(S_T - K)^+$ costs $v(x, K, T, t, \sigma, r_{usd}, r_{eur}, 1)$ US\$ and hence $v(x, K, T, t, \sigma, r_{usd}, r_{eur}, 1)/x$ €. This €-call option can also be viewed as a US\$-put option with payoff $K\left(\frac{1}{K} - \frac{1}{S_T}\right)^+$. This option costs $Kv\left(\frac{1}{x}, \frac{1}{K}, T, t, \sigma, r_{eur}, r_{usd}, -1\right)$ € in Frankfurt, because S_t and $\frac{1}{S_t}$ have the same volatility. Of course, the New York value and the Frankfurt value must agree, which leads to Equation (1.41).

1.4.9 *Euro related symmetries of value, delta and leverage*

Let us now consider the example of S_t modelling the exchange rate £/DM. After the currency Euro has been introduced, we need to know how to relate options written on £/DM to options on €/£. We denote by $E = 1.95583$ the fixed exchange rate €/DM. Then E/S_t serves as model for €/£. Combining the foreign–domestic symmetry Equation (1.41) with the space-homogeneity Equation (1.35) we obtain

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = \frac{Kx}{E} v\left(\frac{E}{x}, \frac{E}{K}, T, t, \sigma, r_f, r_d, -\phi\right) \tag{1.42}$$

Taking the derivative with respect to x on both sides results in

$$\begin{aligned} v_x(x, K, T, t, \sigma, r_d, r_f, \phi) = &\frac{K}{E} v\left(\frac{E}{x}, \frac{E}{K}, T, t, \sigma, r_f, r_d, -\phi\right) \\ &- \frac{K}{x} v_x\left(\frac{E}{x}, \frac{E}{K}, T, t, \sigma, r_f, r_d, -\phi\right) \end{aligned} \tag{1.43}$$

In particular, the deltas of identical options are *not* exactly negatives of each other. This is only approximately correct. The right quantities to compare are not the deltas, but the dimensionless leverages, because Equation (1.43) implies

$$\frac{x v_x(x, K, T, t, \sigma, r_d, r_f, \phi)}{v(x, K, T, t, \sigma, r_d, r_f, \phi)} = 1 - \frac{\frac{E}{x} v_x\left(\frac{E}{x}, \frac{E}{K}, T, t, \sigma, r_f, r_d, -\phi\right)}{v\left(\frac{E}{x}, \frac{E}{K}, T, t, \sigma, r_f, r_d, -\phi\right)} \tag{1.44}$$

This means that the leverages of a £ call and an identical € put add up to one. Note the factor E could be cancelled on the right-hand side to produce a plain foreign–domestic leverage symmetry.

1.5 Quotation

The value of vanilla options may be quoted in various ways, out of which the four most-used quotation methods are

d value in domestic currency (or in pips of the very same),

% d value in % measured in units of the strike,

f value in foreign currency (or in pips of the very same),

% f value in % of foreign currency.

The Black–Scholes formula quotes **d**. The others can be computed using the following instruction

$$\mathbf{d} \xrightarrow{\times\frac{100}{x}} \%\mathbf{f} \xrightarrow{\times\frac{x}{K}} \%\mathbf{d} \xrightarrow{\times\frac{1}{100x}} \mathbf{f} \xrightarrow{\times xK} \mathbf{d} \tag{1.45}$$

1.6 Dual Black–Scholes Partial Differential Equation

The value function for vanilla options can be written as

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = e^{-r_d(T-t)} \mathbb{E}[F|S_t = x] \tag{1.46}$$

Consequently, the process $v(t, S_t)e^{-r_d t} = e^{-r_d T}\mathbb{E}[F|S_t]$ is a martingale, whence the dt-coefficient of its differential must vanish. Therefore $v(x, K, T, t, \sigma, r_d, r_f, \phi)$ satisfies the Black–Scholes PDE

$$v_t - r_d v + (r_d - r_f)xv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0 \tag{1.47}$$

This can easily be remembered by noting that the derivatives have the same sign.

Viewing v as a function of T and K, one can verify by direct computation that the so-called dual Black–Scholes PDE

$$-v_T - r_f v + (r_f - r_d)Kv_K + \frac{1}{2}\sigma^2 K^2 v_{KK} = 0 \tag{1.48}$$

also holds. We note that the Black–Scholes equation holds for all options, whereas its dual is a particularity of put and call options. More details on this issue can be found in Anderson and Brotherton-Ratcliffe [1] and Dupire [9].

1.7 Retrieving the arguments

1.7.1 *Implied volatility*

Since $v_\sigma > 0$, the function $\sigma \mapsto v(x, K, T, t, \sigma, r_d, r_f, \phi)$ is

1. strictly increasing, and also
2. concave up for $\sigma \in (0, \sqrt{2|\ln f - \ln K|/\tau})$,
3. concave down for $\sigma \in (\sqrt{2|\ln f - \ln K|/\tau}, \infty)$,

and also satisfies

$$v(x, K, T, t, \sigma = 0, r_d, r_f, \phi) = [\phi(xe^{-r_f\tau} - Ke^{-r_d\tau})]^+ \tag{1.49}$$

$$v(x, K, T, t, \sigma = \infty, r_d, r_f, \phi = 1) = xe^{-r_f\tau} \tag{1.50}$$

$$v(x, K, T, t, \sigma = \infty, r_d, r_f, \phi = -1) = Ke^{-r_d\tau} \tag{1.51}$$

$$v_\sigma(x, K, T, t, \sigma = 0, r_d, r_f, \phi) = xe^{-r_f\tau}\sqrt{\tau}/\sqrt{2\pi}\mathbb{I}_{\{f=K\}} \tag{1.52}$$

Consequently, there exists a unique implied volatility $\sigma = \sigma(v, x, K, T, t, r_d, r_f, \phi)$ for a given value, v, which can be found by a Newton–Raphson method. However, the starting guess for employing this method should be chosen with care, because the mapping $\sigma \mapsto v(x, K, T, t, \sigma, r_d, r_f, \phi)$ has a saddle point at

$$\left(\sqrt{\frac{2}{\tau}|\ln\frac{f}{K}|}, \phi\left\{xe^{-r_f\tau}\mathcal{N}\left(\phi\sqrt{2\tau\left[\ln\frac{f}{K}\right]^+}\right) - Ke^{-r_d\tau}\mathcal{N}\left(\phi\sqrt{2\tau\left[\ln\frac{K}{f}\right]^+}\right)\right\}\right) \tag{1.53}$$

To ensure convergence of the Newton–Raphson method, we are advised to use initial guesses for σ on the same side of the saddle point as the desired implied volatility. The danger is that a large initial guess could lead to a negative successive guess for σ. Therefore one should start with small initial guesses at or below the saddle point. For at-the-money options, the saddle point is degenerate for a zero volatility and small volatilities serve as good initial guesses.

1.7.2 *Strike given delta*

Since $v_x = \Delta = \phi e^{-r_f \tau} \mathcal{N}(\phi d_+)$ we can retrieve the strike as

$$K = x \exp\left\{-\phi \mathcal{N}^{-1}(\phi \Delta\, e^{r_f \tau}) \sigma \sqrt{\tau} + \sigma \theta_+ \tau\right\} \tag{1.54}$$

1.7.3 *Volatility given delta*

The mapping $\sigma \mapsto \Delta = \phi e^{-r_f \tau} \mathcal{N}(\phi d_+)$ is not one-to-one. Thus using just the delta to retrieve the volatility of an option is not advisable. The two solutions are given by

$$\sigma_\pm = \frac{1}{\sqrt{\tau}} \left\{ \phi \mathcal{N}^{-1}(\phi \Delta\, e^{r_f \tau}) \pm \sqrt{(\mathcal{N}^{-1}(\phi \Delta\, e^{r_f \tau}))^2 - \sigma \sqrt{\tau}(d_+ + d_-)} \right\} \tag{1.55}$$

1.8 Greeks in terms of deltas

Foreign Exchange (FX) markets have adapted to speak about vanilla options in terms of deltas and quote prices in terms of volatility. This makes a ten-delta call a financial object as such independent of spot and strike. This method and the quotation in volatility makes objects and prices transparent in a very intelligent and user-friendly way. At this point we list the Greeks in terms of deltas instead of spot and strike. Let us introduce the quantities

$$\Delta_+ \stackrel{\triangle}{=} \phi e^{-r_f \tau} \mathcal{N}(\phi d_+) \text{ spot delta} \tag{1.56}$$

$$\Delta_- \stackrel{\triangle}{=} -\phi e^{-r_d \tau} \mathcal{N}(\phi d_-) \text{ dual delta} \tag{1.57}$$

which we assume to be given. From these we can retrieve

$$d_+ = \phi \mathcal{N}^{-1}(\phi e^{r_f \tau} \Delta_+) \tag{1.58}$$

$$d_- = \phi \mathcal{N}^{-1}(-\phi e^{r_d \tau} \Delta_-) \tag{1.59}$$

1.8.1 *Interpretation of dual delta*

The dual delta introduced in Equation (1.23) as the sensitivity with respect to strike has another – more practical – interpretation in an FX setup. We have seen in Section 1.4 that the domestic value

$$v(x, K, \tau, \sigma, r_d, r_f, \phi) \tag{1.60}$$

corresponds to a foreign value

$$v\left(\frac{1}{x}, \frac{1}{K}, \tau, \sigma, r_f, r_d, -\phi\right) \tag{1.61}$$

up to an adjustment of the nominal amount by the factor xK. From a foreign viewpoint the delta is thus given by

$$
\begin{aligned}
& -\phi e^{-r_d \tau} \mathcal{N}\left(-\phi \frac{\ln\left(\frac{K}{x}\right) + \left(r_f - r_d + \frac{1}{2}\sigma^2 \tau\right)}{\sigma\sqrt{\tau}} \right) \\
& = -\phi e^{-r_d \tau} \mathcal{N}\left(\phi \frac{\ln\left(\frac{x}{K}\right) + \left(r_d - r_f - \frac{1}{2}\sigma^2 \tau\right)}{\sigma\sqrt{\tau}} \right) \\
& = \Delta_- \qquad (1.62)
\end{aligned}
$$

which means the dual delta is the delta from the foreign viewpoint. We will see below that foreign rho, vega and gamma do not require to know the dual delta. We will now state the Greeks in terms of $x, \Delta_+, \Delta_-, r_d, r_f, \tau, \phi$.

1.8.2 *List of Greeks*

Value

$$v(x, \Delta_+, \Delta_-, r_d, r_f, \tau, \phi) = x\Delta_+ + x\Delta_- \frac{e^{-r_f \tau} n(d_+)}{e^{-r_d \tau} n(d_-)} \qquad (1.63)$$

(Spot) delta

$$\frac{\partial v}{\partial x} = \Delta_+ \qquad (1.64)$$

Forward delta

$$\frac{\partial v}{\partial f} = e^{(r_f - r_d)\tau} \Delta_+ \qquad (1.65)$$

Gamma

$$\frac{\partial^2 v}{\partial x^2} = e^{-r_f \tau} \frac{n(d_+)}{x(d_+ - d_-)} \qquad (1.66)$$

Taking a trader's gamma (change of delta if spot moves by 1%) additionally removes the spot dependence, because

$$\Gamma_{trader} = \frac{x}{100} \frac{\partial^2 v}{\partial x^2} = e^{-r_f \tau} \frac{n(d_+)}{100(d_+ - d_-)} \qquad (1.67)$$

Speed

$$\frac{\partial^3 v}{\partial x^3} = -e^{-r_f \tau} \frac{n(d_+)}{x^2 (d_+ - d_-)^2} (2d_+ - d_-) \qquad (1.68)$$

Theta

$$\frac{1}{x} \frac{\partial v}{\partial t} = -e^{-r_f \tau} \frac{n(d_+)(d_+ - d_-)}{2\tau} + \left[r_f \Delta_+ + r_d \Delta_- \frac{e^{-r_f \tau} n(d_+)}{e^{-r_d \tau} n(d_-)} \right] \qquad (1.69)$$

Charm

$$\frac{\partial^2 v}{\partial x \partial \tau} = -\phi r_f e^{-r_f \tau} \mathcal{N}(\phi d_+) + \phi e^{-r_f \tau} n(d_+) \frac{2(r_d - r_f)\tau - d_-(d_+ - d_-)}{2\tau(d_+ - d_-)} \tag{1.70}$$

Colour

$$\frac{\partial^3 v}{\partial x^2 \partial \tau} = -\frac{e^{-r_f \tau} n(d_+)}{2x\tau(d_+ - d_-)} \left[2r_f \tau + 1 + \frac{2(r_d - r_f)\tau - d_-(d_+ - d_-)}{2\tau(d_+ - d_-)} d_+ \right] \tag{1.71}$$

Vega

$$\frac{\partial v}{\partial \sigma} = x e^{-r_f \tau} \sqrt{\tau} n(d_+) \tag{1.72}$$

Volga

$$\frac{\partial^2 v}{\partial \sigma^2} = x e^{-r_f \tau} \tau n(d_+) \frac{d_+ d_-}{d_+ - d_-} \tag{1.73}$$

Vanna

$$\frac{\partial^2 v}{\partial \sigma \partial x} = -e^{-r_f \tau} n(d_+) \frac{\sqrt{\tau} d_-}{d_+ - d_-} \tag{1.74}$$

Rho

$$\frac{\partial v}{\partial r_d} = -x\tau \Delta_- \frac{e^{-r_f \tau} n(d_+)}{e^{-r_d \tau} n(d_-)} \tag{1.75}$$

$$\frac{\partial v}{\partial r_f} = -x\tau \Delta_+ \tag{1.76}$$

Dual delta

$$\frac{\partial v}{\partial K} = \Delta_- \tag{1.77}$$

Dual gamma

$$\frac{\partial^2 v}{\partial K^2} = \frac{\partial^2 v}{\partial x^2} \tag{1.78}$$

Dual theta

$$\frac{\partial v}{\partial T} = -v_t \tag{1.79}$$

As an important example we consider vega.

1.8.3 *Vega given delta*

The mapping $\Delta \mapsto v_\sigma = x e^{-r_f \tau} \sqrt{\tau} n(\mathcal{N}^{-1}(e^{r_f \tau} \Delta))$ is important for trading vanilla options. Observe that this function does not depend on r_d or σ, just on r_f. Quoting vega in % foreign will additionally remove the spot dependence. This means that for a moderately stable foreign term structure curve, traders will be able to use a moderately stable vega matrix. That is, for $r_f = 3\%$ the vega matrix looks like this:

Mat/Δ	50%	45%	40%	35%	30%	25%	20%	15%	10%	5%
1D	2	2	2	2	2	2	1	1	1	1
1W	6	5	5	5	5	4	4	3	2	1
1W	8	8	8	7	7	6	5	5	3	2
1M	11	11	11	11	10	9	8	7	5	3
2M	16	16	16	15	14	13	11	9	7	4
3M	20	20	19	18	17	16	14	12	9	5
6M	28	28	27	26	24	22	20	16	12	7
9M	34	34	33	32	30	27	24	20	15	9
1Y	39	39	38	36	34	31	28	23	17	10
2Y	53	53	52	50	48	44	39	32	24	14
3Y	63	63	62	60	57	53	47	39	30	18

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